Your Exam Manager will receive a copy of the 2015 AIME Solution Pamphlet with the scores.

CONTACT US – *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

MAA American Mathematics Competitions
9050 Junction Drive
Annapolis Junction, MD 20701
Phone: 800.527.3690; Fax: 240.396.5647; email: amcinfo@maa.org

The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Jonathan M. Kane
AIME Chair
kanej@uww.edu

2015 USA(J)MO – THE USA MATHEMATICAL OLYMPIAD (USAMO) and the USA MATHEMATICAL JUNIOR OLYMPIAD (USAJMO) are each a 6-question, 9-hour, essay-type examination. The USA(J)MO will be held in your school on Tuesday and Wednesday, April 28 & 29, 2015. Your teacher has more details on who qualifies for the USA(J)MO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USA(J)MO is to study previous years of these exams. Copies may be ordered from the web site indicated below.

PUBLICATIONS – For a complete listing of available publications please visit the MAA Bookstore or Competitions site at maa.org.

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American Mathematics Competitions

33rd Annual

AIME II

American Invitational Mathematics Examination I
Wednesday, March 25, 2015

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 12 are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of the AIME and the American Mathematics Contest 10 are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO & the USAJMO will be given in your school on TUESDAY and WEDNESDAY, April 28 & 29, 2015.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Let N be the least positive integer that is both 22 percent less than one integer and 16 percent greater than another integer. Find the remainder when N is divided by 1000.

2. In a new school 40 percent of the students are freshmen, 30 percent are sophomores, 20 percent are juniors, and 10 percent are seniors. All freshmen are required to take Latin, and 80 percent of the sophomores, 50 percent of the juniors, and 20 percent of the seniors elect to take Latin. The probability that a randomly chosen Latin student is a sophomore is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

3. Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m.

4. In an isosceles trapezoid, the parallel bases have lengths $\log 3$ and $\log 192$, and the altitude to these bases has length $\log 16$. The perimeter of the trapezoid can be written in the form $\log 2^p 3^q$, where p and q are positive integers. Find $p+q$.

5. Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

6. Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers a and c. Can you tell me the values of a and c?"

   After some calculations, Jon says, "There is more than one such polynomial."

   Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?"

   Jon says, "There are still two possible values of c."

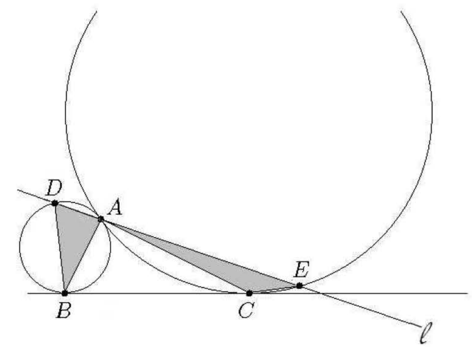
   Find the sum of the two possible values of c.

7. Triangle ABC has side lengths $AB = 12$, $BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial

   $$\text{Area}(PQRS) = \alpha w - \beta \cdot w^2.$$

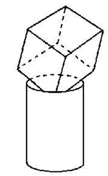
   Then the coefficient $\beta = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

15. Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ so that line BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ, and the areas of $\triangle DBA$ and $\triangle ACE$ are equal. This common area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



8. Let a and b be positive integers satisfying $\frac{ab+1}{a+b} < \frac{3}{2}$. The maximum possible value of $\frac{a^3b^3+1}{a^3+b^3}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

9. A cylindrical barrel with radius 4 feet and height 10 feet is full of water. A solid cube with side length 8 feet is set into the barrel so that the diagonal of the cube is vertical. The volume of water thus displaced is v cubic feet. Find v^2.



10. Call a permutation $a_1, a_2, \ldots, a_n$ of the integers $1, 2, \ldots, n$ *quasi-increasing* if $a_k \leq a_{k+1} + 2$ for each $1 \leq k \leq n-1$. For example, 53421 and 14253 are quasi-increasing permutations of the integers $1, 2, 3, 4, 5$, but 45123 is not. Find the number of quasi-increasing permutations of the integers $1, 2, \ldots, 7$.

11. The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

12. There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

13. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = \sum_{k=1}^{n} \sin(k)$, where k represents radian measure. Find the index of the 100th term for which $a_n < 0$.

14. Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.